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Similarity Solution and Lie Symmetry for a Coupled Nonlinear System

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The Lie point symmetries of a set of coupled nonlinear partial differential equations are considered. The system is an extended version of the usual nonlinear Schrödinger equation. In the similarity variable deduced from the symmetry analysis, the system is equivalent to the Painlevé III in Ince's classification. By starting from a solution of the Painlevé equation, one can reproduce various classes of solutions of the original PDEs. Such solutions include both rational and progressive types or a combination of the two.

1. INTRODUCTION

The symmetry approach (Ibragimov, 1978) to nonlinear partial differential equations plays a prominent role among the various approaches developed for their analysis. An interesting outcome of such an analysis is the reduction of the nonlinear equations to an ordinary nonlinear equation which may belong to the category of equations studied long ago by Painlevé (Ince, 1944). Here we consider the symmetry properties of an extended set of nonlinear Schrödinger equations, which frequently occurs in plasma physics and hydrodynamics (Petviashvilli, 1981). The similarity variable is deduced, and with its help these equations are shown to be equivalent to the Painlevé III class in Ince's (1944) scheme. After this we proceed to the actual construction of the solutions of the original PDSs from those of the Painlevé. It is then observed that a Backlund transformation due to Airault helps to construct rational or progressive solutions or a solution that is a combination of both.

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2. FORMULATION

The set of nonlinear equations under consideration reads

$$\begin{aligned} iu_t &= vu + u_{xx} + \gamma\delta\lambda^2|u|^2u \\ vt + \alpha\beta\lambda|u|_x^2 &= 0 \end{aligned} \tag{1}$$

When v is set to zero, these are nothing but the equations of NLSE. Let us now set

$$u = u^0 + iu'$$

Then demand the invariance of the resulting equations under the Lie point transformation,

$$\begin{aligned} x &\to x + \varepsilon\xi(x, t, u^0, u', v) \\ t &\to t + \varepsilon\tau(x, t, u^0, u', v) \\ u^0 &\to u^0 + \varepsilon\sigma^0(x, t, u^0, u', v) \\ u' &\to u' + \varepsilon\sigma'(x, t, u^0, u', v) \\ v &\to v + \varepsilon\eta(x, t, u^0, u', v) \end{aligned} \tag{2}$$

leading to the following equations for the transformation functions:

$$\sigma_t^0 = u'\eta + v\sigma' + \sigma'_{xx} + \gamma\lambda^2\delta[(u^0)^2\sigma' + 2u^0u'\sigma^0 + 3\sigma'(u')^2] \tag{3a}$$

$$-\sigma'_t = u^0\eta + v\sigma^0 + \sigma^0_{xx} + \gamma\delta\lambda^2[(u_1)^2\sigma^0 + 2u^0u'\sigma' + 3\sigma^0(u^0)^2] \tag{3b}$$

$$\eta_t + 2\alpha\beta\lambda[\sigma^0u_x^0 + u^0\sigma_x^0 + u'\sigma'_x + \sigma'u'_x] = 0 \tag{3c}$$

The derivatives of the functions ξ, τ, σ^0, σ', and η occurring in (3a)-(3c) are to be interpreted as total derivatives. Such formulas for total derivatives can be constructed following the procedure outlined in Bluman and Cole (1974). We do not reproduce these here, as they are quite straightforward (but laborious) to deduce. Equating to zero the various coefficients of u^0, u', v, and their derivatives, we get

$$\frac{\partial\sigma'}{\partial u_0} + \frac{\partial\sigma^0}{\partial u'} = 0$$

$$\frac{\partial\sigma'}{\partial u'} - 2\frac{\partial\xi}{\partial x} - \frac{\partial\sigma^0}{\partial u^0} + \frac{\partial\tau}{\partial t} = 0$$

$$2\frac{\partial^2\sigma'}{\partial x\,\partial u_0} - 2\alpha\beta\lambda\frac{\partial\sigma^0}{\partial v}u^0 + \frac{\partial\xi}{\partial t} = 0$$

$$2\frac{\partial^2\sigma'}{\partial x\,\partial u'} - \frac{\partial^2\xi}{\partial x^2} + 2\alpha\beta\lambda\frac{\partial\sigma^0}{\partial v}u' = 0$$

$$\frac{\partial^2\sigma'}{\partial x\,\partial v}=0; \qquad \frac{\partial\sigma'}{\partial v}=0$$

$$-\frac{\partial\sigma^0}{\partial t}+\frac{\partial\sigma^0}{\partial u'}\{u^0 v+\gamma\delta\lambda^2[(u^0)^3+u^0(u')^2]\}$$

$$+\left(\frac{\partial\tau}{\partial t}-\frac{\partial\sigma^0}{\partial u^0}\right)\{u'v+\gamma\delta\lambda^2[(u^0)^2u'+(u')^3]\}$$

$$+v\sigma'+u'\eta+\frac{\partial^2\sigma'}{\partial x^2}+\gamma\delta\lambda^2\{[(u^0)^2+3(u')^2]\sigma'+2u^0u'\sigma_0\}=0 \tag{4a}$$

$$\frac{\partial\tau}{\partial t}-\frac{\partial\sigma'}{\partial u'}+\frac{\partial\sigma^0}{\partial u^0}-2\frac{\partial\xi}{\partial x}=0\frac{\partial\sigma'}{\partial u^0}+\frac{\partial\sigma^0}{\partial u'}=0; \qquad \frac{\partial\sigma^0}{\partial v}=0; \qquad \frac{\partial^2\sigma^0}{\partial x\,\partial v}=0$$

$$-2\alpha\beta\lambda\frac{\partial\sigma'}{\partial v}u^0+2\frac{\partial^2\sigma^0}{\partial x\,\partial u^0}-\frac{\partial^2\xi}{\partial x^2}=0$$

$$-2\alpha\beta\lambda\frac{\partial\sigma'}{\partial v}u'-\frac{\partial\xi}{\partial t}+2\frac{\partial^2\sigma_0}{\partial x\,\partial u'}=0$$

$$\frac{\partial\sigma'}{\partial t}+\frac{\partial\sigma'}{\partial u^0}\{u'v+\gamma\delta\lambda^2[(u^0)^2u'+u^0(u')^2]\}$$

$$+v\sigma^0+u^0\eta+\left(\frac{\partial\tau}{\partial t}-\frac{\partial\sigma'}{\partial u'}\right)\{u^0v+\gamma\lambda^2\delta[(u^0)^3$$

$$+u^0(u')^2]\}+\gamma\lambda^2\delta\{[3(u^0)^2+(u')^2]\gamma^0+2u^0u'\sigma'\}=0$$

and finally

$$\frac{\partial\eta}{\partial u'}=0=\frac{\partial\eta}{\partial u^0} \tag{4b}$$

$$\frac{\partial\xi}{\partial t}=2\alpha\beta\lambda\left(u^0\frac{\partial\sigma^0}{\partial v}+u'\frac{\partial\sigma'}{\partial v}\right)$$

$$u^0\left(\frac{\partial\tau}{\partial t}-\frac{\partial\eta}{\partial v}\right)+\sigma^0+u^0\frac{\partial\sigma^0}{\partial u^0}+u'\frac{\partial\sigma'}{\partial u^0}-u^0\frac{\partial\xi}{\partial x}=0$$

$$u'\left(\frac{\partial\tau}{\partial t}-\frac{\partial\eta}{\partial v}\right)+\sigma'+u^0\frac{\partial\sigma^0}{\partial u'}+u'\frac{\partial\sigma'}{\partial u'}-u'\frac{\partial\xi}{\partial x}=0$$

$$\frac{\partial\eta}{\partial t}+2\alpha\beta\lambda\left(u^0\frac{\partial\sigma^0}{\partial x}+u'\frac{\partial\sigma'}{\partial x}\right)=0 \tag{4c}$$

respectively, from (3a)-(3c). Due to the large number of equations, the solution set is quite simple and is given as

$$\begin{aligned}
\sigma^0 &= (\alpha'+\beta' t)u' \\
\sigma' &= -(\alpha'+\beta' t)u^0 \\
\eta &= \beta' \text{ (constant)} \\
\xi &= \delta' \text{ (constant)} \\
\tau &= \tau \text{ (constant)}
\end{aligned} \tag{5}$$

It is interesting to note that in the limit when the second field $v \to 0$ (also for consistency $\beta \to 0$) these symmetries reproduce those of ordinary NLSE as given in Roy Chowdury and Paul (1984). The Lagrange equation corresponding to (5) is

$$\frac{du^0}{(\alpha'+\beta' t)u'} = \frac{du'}{-(\alpha'+\beta' t)u^0} = \frac{dv}{\beta'} = \frac{dx}{\delta'} = \frac{dt}{\tau} \tag{6}$$

Integrating these pairwise, we obtain following similarity form of solution

$$\begin{aligned}
u^0 &= C_2^{1/2} \sin\frac{1}{\tau}\left(\alpha' t + \frac{\beta'}{2} t^2 + c_3\right) \\
u' &= c_2^{1/2} \cos\frac{1}{\tau}\left(\alpha' t + \frac{\beta'}{2} t^2 + c_3\right) \\
v &= \frac{\beta'}{\tau} t + \frac{\mathscr{G}(c)}{\tau}
\end{aligned} \tag{7}$$

where $c_1 = \mathscr{G}(c)$, $c_2 = F(c)$, and $c_3 = H(c)$ are arbitrary functions of $c = \tau x - \delta t'$.

Substituting these back in the original system, we get

$$\frac{\partial H}{\partial c} = \frac{F\sigma' - K}{2\tau F} \tag{8a}$$

and

$$\mathscr{G}(c) = \frac{\beta'}{\delta'} c + \frac{\alpha\beta\lambda\tau^2}{\delta'} F(c) + k_1 \tag{8b}$$

where $F(c)$ satisfies

$$F'' = \frac{F'^2}{2F} - \frac{\lambda_3}{2F} + \frac{\lambda_2 F^2}{2} - \frac{c\nu}{2} F - \lambda_1 \frac{F}{2} \tag{8c}$$

with

$$\begin{aligned}\lambda_1 &= (4/\tau^3)(-\alpha'+\delta'^2/4\tau+\tau K_1)\\ \lambda_2 &= -4(\alpha\beta\lambda/\delta'+\gamma\delta\lambda^2/\tau^2)\\ \lambda_3 &= -K_2/\tau^4\\ \nu &= 4\beta'/\tau^2\delta'\end{aligned} \tag{8d}$$

and

$$F''=\frac{\partial^2 F}{\partial c^2}; \qquad F'=\frac{\partial F}{\partial c}$$

Now choosing $c\nu+\lambda_1=y$ and $\lambda_3=\nu^2=\frac{1}{2}$, we get

$$\frac{\partial^2 F}{\partial y^2}=\frac{1}{2F}\left(\frac{\partial F}{\partial y}\right)^2+\lambda_2 F^2-yF-\frac{1}{2F} \tag{9}$$

which is nothing but the Painlevé equation of type III in the category XXXIV of Ince. It is now a standard procedure to set

$$\lambda_2 F/2 = Wy+W^2+y/2 \tag{10}$$

so that W satisfies

$$\partial^2 W/\partial y^2 = 2W^3+yW-(\lambda_2+1/2 \tag{11}$$

Now an equation of this form has been discussed in detail by Airault (1979), in which $-(\lambda_2+1)/2=\delta$ is considered for both δ integer and half-integer. It was shown how one can generate different solutions of (11) with the help of the Backlund transformation. It is essentially proved that if an nth solution of equation (11) can be written as

$$y_n = u'_{n-1}/u_{n-1} - u'_n/u_n \tag{12}$$

where the function u_n satisfies the recurrence formulas

$$u_{n+1}u_{n-1}=(29n+z)u_n^2$$

with q_n given as

$$q_n=-2\frac{d^2}{dz^2\log u_n} \tag{14}$$

So that one can start to construct a series of independent rational solutions of (1) with the help of equations (12)-(14). When $\delta = n$ (integer), such solutions are

$$u_0 = 1, \qquad u_1 = z, \qquad u_2 = z^3 + 4, \ldots \tag{15}$$

It is now easy to observe that if we assume that $(\lambda_2 + 1)/2$ is an integer and as a simplest case let $\lambda_2 = 1$, then (11) has the solution

$$w = 1/y$$

so that $F = y = c\nu + \lambda_1$, and from equation (8a) we get

$$H = \frac{\delta'}{2\tau}c - \frac{k}{2\tau\nu}\log(c\nu + \lambda_1) + k_2$$

So we get finally the first set of similarity solutions in the form

$$u = i\left(\frac{\tau x - \delta' t}{\sqrt{2}} + \lambda_1\right)^{1/2} \exp\left(-\frac{iz}{\tau}\right)$$

$$z = \alpha' t + \frac{\beta'}{2}t^2 + \frac{\delta'}{2\tau}(\tau x - \delta' t) + K_2 - \frac{i\tau}{2}\log\left(\frac{\tau x - \delta' t}{\sqrt{2}} + \lambda_1\right) \tag{16}$$

$$v = \frac{1}{\tau}\left[\beta' t + \frac{\beta'}{\delta'}(\tau x + \delta' t) + \frac{\alpha\beta\lambda}{\delta'}\tau^2\left(\frac{\tau x - \delta' t}{\sqrt{2}} + \lambda_1\right) + K_1\right] \tag{17}$$

A second set of solutions may be constructed if we assume $(\lambda_2 + 1)/2 = -1$, so that $\lambda_2 = -3$. With this value, (11) has the solution

$$W = -1/y$$

so that

$$F = -\frac{2}{3}\left(\frac{2}{y^2} + \frac{y}{2}\right) \tag{18}$$

Proceeding as before, we can again determine H and $\mathcal{G}$; finally, the solution set is of the form

$$\begin{aligned} u &= -\left\{\frac{4 + [(\tau x - \delta' t)/\sqrt{2} + \lambda_1]^3}{3[(\tau x - \delta' t)/\sqrt{2} + \lambda_1]^2}\right\}^{1/2} \exp\left(-\frac{i'}{\tau}z\right) \\ z &= \alpha' t + \frac{\beta'}{2}t^2 + \frac{\delta'}{2\tau}(\tau x - \delta' t) + \frac{i\tau}{2}\left[4 + \left(\frac{\tau x - \delta' t}{\sqrt{2}} + \lambda_1\right)^3\right] + K_3 \end{aligned} \tag{19}$$

$$v \quad = \frac{1}{\tau}\left[\beta' t + \frac{\beta'}{\delta'}(\tau x - \delta' t) - \frac{\alpha\beta\lambda}{\delta'}\tau^2 \frac{4 + [(\tau x - \delta' t)/\sqrt{2} + \lambda_1]^3}{3[(\tau x - \delta' t)/\sqrt{2} + \lambda_1]^2} + K_1\right] \qquad (20)$$

3. DISCUSSIONS

In the above computation we have extended the Lie symmetry approach to a coupled system of NLSE and another evolution equation. Such an equation is of importance in plasma and other physical phenomena. It is interesting to observe that the system reduces to Painlevé III with a rich structure of rational and similarity solutions. It is quite clear that starting from the Backlund transformation of Airlaut for P III, one can construct a BT for the original set of equations. But until now no inverse scattering is known for this equation. Hence, it will be interesting to test the complete integrability of (1) by other means.

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